Richtmyer-Meshkov flow in elastic solids

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Richtmyer-Meshkov flow is studied by means of an analytical model which describes the asymptotic oscillations of a corrugated interface between two perfectly elastic solids after the interaction with a shock wave. The model shows that the flow stability is due to the restoring effect of the elastic force. It provides a simple approximate but still very accurate formula for the oscillation period. It also shows that as it is observed in numerical simulations, the amplitude oscillates around a mean value equal to the post-shock amplitude, and that this is a consequence of the stress free conditions of the material immediately after the shock interaction. Extensive numerical simulations are presented to validate the model results.

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One of the goals of the FAIR (Facility for Antiproton and Ion Research) facility to be constructed at GSI Darmstadt is the research on high energy density physics states of the matter created in the laboratory [1,2]. In particular the LAPLAS (Laboratory of Planetary Sciences) experiment is being designed in order to study equation of state and transport properties of high energy density matter by using a low entropy cylindrical implosion driven by an intense heavy ion beam that compresses a test material as, for example, hydrogen [3–8]. The implosion stability is one issue that is being addressed at present [9,10]. Numerical simulations show that during the implosion the pusher remains in solid state thus retaining the elastic and plastic properties of the material and affecting the development of the instabilities [11,12]. Recently we have studied the Rayleigh-Taylor (RT) instability of a perfectly elastic semi-infinite material by using a simple but still very accurate model that yields the instability evolution [9].

Besides, a Richtmyer-Meshkov (RM)-like instability may occur when a shock is launched into the pusher from the absorber-pusher interface [13,14] and it may also happen when this shock arrives at the pusher-hydrogen interface [15–22]. This may be another issue of possible concern that must be taken into account since RM flow will set the initial conditions for the later development of the RT instability. Recently, Plohr and Plohr [20] have presented linear numerical simulations of the RM flow in elastic materials. As it is noted by these authors, the term "flow" is more appropriate than "instability" in the case of elastic materials since, in contrast to the classical gas dynamical case, the interface remains oscillating stably. The situation seems to be similar to the case of RM-like flow in ablation fronts [13,14] or in an interface between two inviscid fluids with surface tension [19]. In those cases a restoring force exists that prevents the development of the instability.

RM instability is a well known phenomenon in gas dynamics that has been extensively studied theoretically, experimentally, and by means of numerical simulations for more than forty years [13–22]. However, a complete exact analytical theory has not been available until very recently [16,17]. Such a theory has shown that the largely used "impulsive model" proposed originally by Richtmyer [21] and frequently used in their different versions is actually inappropriate for the description of the instability. This is essentially because, in an attempt to assimilate the problem to that of the RT instability, such a model assumes an impulsive "gravity" acceleration across the interface, together with the assumptions of incompressibility and irrotationality of the velocity perturbations. The detailed study of RM instability performed by Wouchuk [16,17] has shown that even for relatively weak incident shocks, for which the impulsive model is supposed to yield the best results, the vorticity generated at the interface by the shocks is important in determining the instability growth rate. For strong shocks the bulk vorticity behind the transmitted shock must also be taken into account so that the fluid between the interface and the shock can never be considered as irrotational. The previous facts determine the initial velocity of the interface as well as the asymptotic growth rate achieved after an initial transient phase that cannot be described by the impulsive model. Thus although the basic concept that asymptotically there are no forces acting on the interface remains valid and, in the classical case, the perturbation amplitude grows with constant velocity [15–18], it cannot be used to determine the initial conditions as it was done by Richtmyer.

Nevertheless, we can still use this fact and assume that asymptotically, in an elastic solid, the only force acting on the interface is the restoring elastic force. Here we will use the simple method presented in Ref. [9] for obtaining an approximate analytical formula for the interface oscillation

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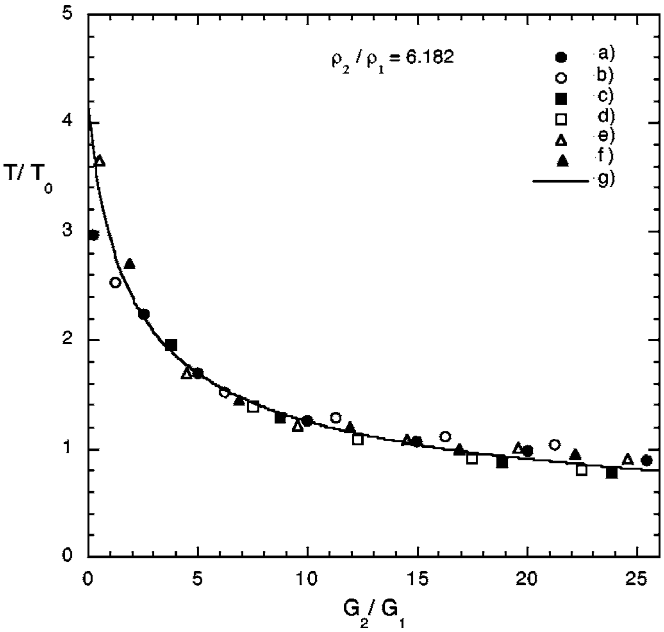

FIG. 1. Dimensionless period T/T_0 as a function of the shear moduli ratio G_2/G_1 for $\rho_2/\rho_1=6.182$ and different cases: (a) $G_1=27.1$ GPa, $\rho_1=2.7$ g/cm^3, $\lambda=2.5$ mm; (b) $G_1=54$ GPa, $\rho_1=2.7$ g/cm^3, $\lambda=2.5$ mm; (c) $G_2=35$ GPa, $\rho_1=2.7$ g/cm^3, $\lambda=2.5$ mm; (d) $G_2=69$ GPa, $\rho_1=2.7$ g/cm^3, $\lambda=2.5$ mm; (e) $G_1=27.1$ GPa, $\rho_1=4.05$ g/cm^3, $\lambda=2.5$ mm; (f) $G_1=27.1$ GPa, $\rho_1=2.7$ g/cm^3, $\lambda=5$ mm; (g) Eq. (13).

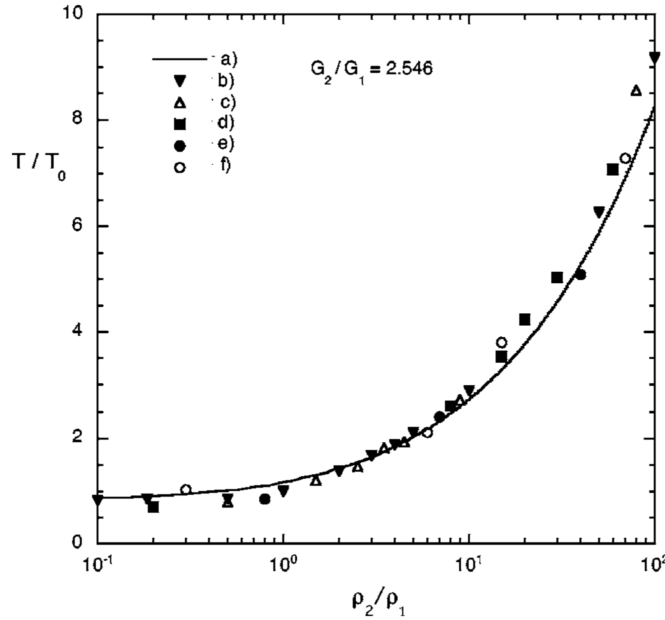

FIG. 2. Dimensionless period T/T_0 as a function of the densities ratio ρ_2/ρ_1 for $G_2/G_1=2.546$ and different cases: (a) Eq. (13); (b) $\rho_1=2.7$ g/cm^3, $G_1=27.1$ GPa, $\lambda=2.5$ mm; (c) $\rho_1=5.4$ g/cm^3, $G_1=27.1$ GPa, $\lambda=2.5$ mm; (d) $\rho_1=48.6$ g/cm^3, $G_1=27.1$ GPa, $\lambda=2.5$ mm; (e) $\rho_1=5.4$ g/cm^3, $G_1=27.1$ GPa, $\lambda=5$ mm; (f) $\rho_1=2.7$ g/cm^3, $G_1=54.2$ GPa, $\lambda=2.5$ mm.

frequency in terms of the material parameters (densities ρ_1 and ρ_2, and shear moduli G_1 and G_2) and of the perturbation wave number $k=2\pi/\lambda$ (λ is the perturbation wavelength). In Ref. [9] we have shown how to calculate the magnitude of the force on a perturbed interface due to a perfectly elastic medium (a Hookean solid). In the case of a Hookean solid the constitutive model for the material is [23,24]

$$\frac{\partial S_{ij}}{\partial t}=2GD_{ij},\quad D_{ij}=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right),\tag{1}$$

where G is the shear modulus of the material, S_{ij} is the deviatoric part of the stress tensor, i and j denote the coordinate directions ($i,j=x,y,z$), and D_{ij} is the strain tensor. Then, the force $f_i^{(\nu)}$ per unit of area that acts on the interface because of each one of the materials ($\nu=1,2$) is [24]

$$f_i^{(\nu)}=p_\nu n_i^{(\nu)}-S_{ij}^{(\nu)}n_j^{(\nu)},\tag{2}$$

where p_ν is the pressure in the material ν and n_i is the component i of the unit vector $\mathbf{n}^{(2)}$ directed outwards along the normal to the interface. Considering two dimensional perturbations ($i=y,j=x$) and taking the interface normal to the y axis, the force normal to the interface is

$$f_y^{(\nu)}=p_\nu n_y^{(\nu)}-S_{yy}^{(\nu)}n_y^{(\nu)},\tag{3}$$

where we have neglected the term $S_{yx}^{(\nu)}n_x^{(\nu)}$ since in the linear regime it is $n_x^{(\nu)}\sim k\xi\ll 1$ ($n_y^{(\nu)}\approx\pm 1$). Since $p_1=p_2$ and taking the material "1" on the side of the positive "y" and the material "2" on the negative side (the interface is at $y=0$), we have $n_y^{(2)}=-n_y^{(1)}=1$, and the total force on the interface due to both materials turns out:

$$\frac{\partial F_y}{\partial t}=\left(\frac{\partial f_y^{(1)}}{\partial t}+\frac{\partial f_y^{(2)}}{\partial t}\right)A=2G_1\frac{\partial v_y^{(1)}}{\partial y}A-2G_2\frac{\partial v_y^{(2)}}{\partial y}A,\tag{4}$$

where A is the area of the interface. In order to estimate the force, we assume a perturbed velocity field of the form

$$v_y^{(1)}\propto e^{ikx-qy};\quad v_y^{(2)}\propto e^{ikx+qy},\tag{5}$$

which decays from the interface with a characteristic length $q^{-1}=\alpha k^{-1}$, and we choose α as the value that best fits our numerical simulations [9,10,25–28]. Then, it results

$$\frac{\partial F_y}{\partial t}=-2(G_1+G_2)\frac{k}{\alpha}\dot{\xi}A,\tag{6}$$

where we have taken

$$v_y^{(1)}(y=0)=v_y^{(2)}(y=0)=\dot{\xi}.\tag{7}$$

By integrating Eq. (6) and assuming an initially stress free material, we obtain the total force acting on the interface due to both materials:

$$F_y=-2(G_1+G_2)\frac{k}{\alpha}(\xi-\xi_{0*}),\tag{8}$$

where ξ_{0*} is the initial amplitude before the material starts to be deformed by the eventual development of the instability. In the present case it must be taken as the post-interaction value since the transit time of the incident shock is so short that there is not enough time for the material to develop

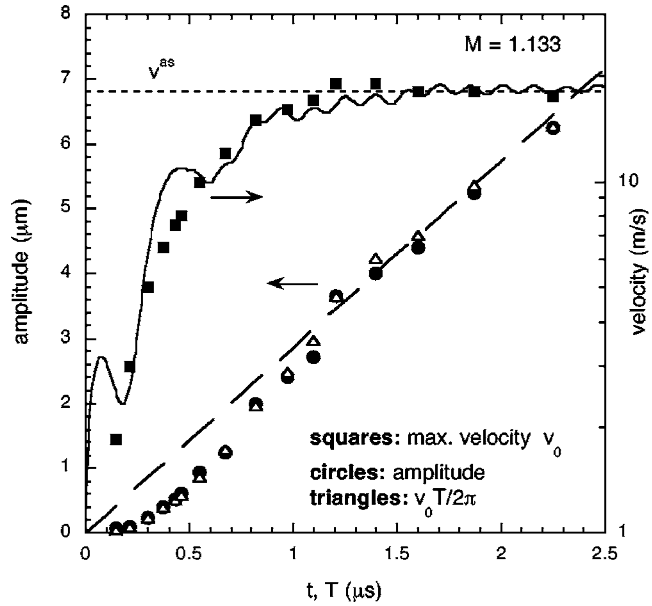

FIG. 3. Maximum oscillation velocity v_0 (squares), oscillation amplitude (circles), and the relationship $v_0T/2\pi$ (triangles) as a function of the oscillation period T. The dashed line is the relationship $v_0^{as}T/2\pi$ as a function of the period T; the full line is the perturbation velocity v_{cl} for the classical case ($G_\nu=0$) as a function of time, and the dotted line indicates the asymptotic classical velocity v^{as}.

stresses. This post-shock initial amplitude can be related with the initial perturbation ξ_0 on the interface as in Refs. [15–18]:

$$\xi_{0*} = \xi_0\left(1 - \frac{v}{u_i}\right), \tag{9}$$

where u_i is the velocity of the incident shock and v is the velocity gained by the interface, originally at rest, in the interaction with the shock.

Since we are dealing with surface modes that decay exponentially from the interface [see Eq. (5)] with the characteristic length $q^{-1}=\alpha k^{-1}$, we find that the mass of both materials involved in the motion is

$$m = m_1 + m_2 = \rho_1\frac{\alpha A}{k} + \rho_2\frac{\alpha A}{k}, \tag{10}$$

and the equation of motion of the interface reads

$$\alpha^2(\rho_1+\rho_2)\frac{\ddot{\xi}}{k} = -2(G_1+G_2)k(\xi - \xi_{0*}), \tag{11}$$

or

$$\ddot{\xi} = -\omega^2(\xi - \xi_{0*}); \quad \omega = \frac{2\pi}{T} = \frac{k}{\alpha}\sqrt{\frac{2(G_1+G_2)}{\rho_1+\rho_2}}, \tag{12}$$

where T is the oscillation period. We can see that for the particular case $G_1 \propto G_2$ considered in Ref. [20] it turns out that $T \sim G_{1,2}^{-1/2}$ such as it was reported as a result of linear simulations. In particular, for the choice $\alpha=1.55$ we obtain an excellent quantitative agreement with those simulations, so that hereafter we will adopt this fitting value. The previous equation can be re-written in a more suitable form for comparison with our numerical simulations:

$$\frac{T}{T_0} = 1.55\sqrt{\frac{1+\rho_2/\rho_1}{2(1+G_2/G_1)}}; \quad T_0 = \frac{2\pi}{k}\sqrt{\frac{\rho_1}{G_1}}. \tag{13}$$

On the other hand, Eq. (12) can be easily integrated with the initial conditions $\dot{\xi}(t=0)=\xi_{0*}$ and $\dot{\xi}(t=0)=v_0$ to yield

$$\xi = \xi_{0*} + \frac{v_0}{\omega}\sin\omega t, \tag{14}$$

where $t=0$ is intended to be the instant after the interaction. v_0 is some initial velocity that cannot be calculated from the present analysis and any prescription for it would be equivalent to the prescriptions used in the impulsive model [21]. Actually the calculation of v_0 would require a self-consistent treatment like that presented in Refs. [15–17]. Nevertheless, Eq. (14) is useful as it shows that oscillations must take place around a mean value equal to ξ_{0*} that is lower than the initial perturbation amplitude ξ_0 [see Eq. (9)], in agreement with the results of the numerical simulations [20]. This is a distinctive fact of the RM flow in elastic materials that is not observed in gas dynamics when restoring forces are present as a consequence, for instance, of ablation [13,14] or surface tension [19]. In those cases oscillations are observed to occur around a mean value equal to zero. Equations (12) and (14) show that in elastic materials the different behavior is caused by the stress free conditions existing a $t=0$, immediately after the interaction of the incident shock with the interface. As we have already mentioned, such a condition is preserved during the interaction because of its very short duration.

We have performed extensive two-dimensional numerical simulations using the finite elements code ABAQUS [29]. For the material equations of state (EOS) we have adopted a Mie-Grüneisen EOS with a Grüneisen coefficient Γ_0 and, in order to express the Hugoniot of the materials, we have taken the usual linear relationship between the shock velocity u_s and the particle velocity v_p, $u_s=c_0+sv_p$, where s and c_0 are parameters characteristic of the material. For the numerical calculations we have considered an incident shock inside fluid "1" driven by a pressure p_s that hits the interface at $y=0$ and then a transmitted shock and a reflected front (shock or rarefaction) are formed. For the EOS of fluid "2" we have taken $s_2=1.67$, $c_{02}=100$ m/s, and $\Gamma_{02}=1.67$. For the EOS of fluid "1" we have $s_1=1.337$, $c_{01}=5380$ m/s, and $\Gamma_{01}=1.97$. Changes in these parameters do not produce any effect on the results[20]. In Fig. 1 we have represented the dimensionless period T/T_0 as a function of the ratio G_2/G_1 between the material shear moduli for the case $\rho_2/\rho_1=6.182$ and varying the rest of the parameters that appear in Eq. (13). As we can see there is an excellent agreement between Eq. (13) and the simulation results. In Fig. 2 the dimensionless period T/T_0 has been represented as a function of the ratio ρ_2/ρ_1 for $G_2/G_1=2.546$ and, again, we have varied the rest of the parameters. In this figure the cases for which $\rho_2/\rho_1<1$ correspond to the situation in which the reflected front is a rarefaction and for $\rho_2/\rho_1>1$ a shock is reflected. We see that in both cases there is a very good agreement between the simulations and the analytic model.

The simulations reported in Figs. 1 and 2 have been obtained for three different shock pressures p_s: 1, 10, and 100 GPa, respectively and we find that the results are independent of the shock intensity such as was reported in Ref. [20]. As it was already noted in Ref. [20], the oscillation frequency is the same in all the cases. Instead, the mean value around which the oscillations take place decreases with the shock intensity according with Eqs. (9) and (14) showing that, in spite of the fact that the oscillation amplitude increases, it always remains lower than the initial value ξ_0.

We have also performed numerical simulations to check the validity of Eq. (14) farther. Results in Fig. 3 correspond to the particular case of a Mach number $M=1.133$ but similar results are obtained in other cases. In this figure we show the maximum oscillation velocity v_0 (squares) and the oscillation amplitude (circles) as a function of the oscillation period T. Triangles represent $v_0T/2\pi$ showing that the oscillation amplitude is given by this relationship in agreement with Eq. (14). As we can appreciate the maximum velocity v_0 achieves an asymptotic value v^{as} for relatively long periods T and, for these cases, the oscillation amplitude grows linearly with T. In the same figure we show the instantaneous velocity of the perturbations v_{cl} for the classical case ($G_\nu=0$) as a function of time. As we can see, the initial velocity v_0 approximately coincides with the instantaneous velocity of the perturbations at the time $t=T$: $v_0 \approx v_{cl}(t=T)$. That is, the initial velocity v_0 to be used in Eq. (14) is just the instantaneous velocity such as it is given by the theory by Wouchuk [16,17] for the classical case, calculated at a time equal to the oscillation period T. In other words, for times shorter than the characteristic time T the effects of the elastic properties of the material are not felt and the instability starts to grow classically. At a time of the order of the period T the elastic effects are felt and the classical perturbation velocity v_{cl} at $t=T$ is taken as the initial velocity for the later evolution of the instability. As we can see, for relatively short periods, the initial velocity corresponds to the value of the classical velocity during the initial transient phase before the asymptotic regime is reached. Therefore the calculation of the initial velocity using the Richtmyer prescription would be in error even in those cases for which such a prescription may give a good approximation for the asymptotic growth rate. A similar case may happen in other RM-like flows as, for instance, in the one that develops in ablation fronts [13,14]. In such a case, the characteristic time associated to the stabilizing ablation process will determine the initial perturbation velocity required to calculate the correct oscillation amplitude.

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